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Investigating future primary teachers' grasping of situations related to unequal partition word problems

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Abstract : This contribution focuses on future primary teachers' grasping of situations related to unequal partition word problems. In the first part of the text we introduce an educational tool called Concept Cartoons, and investigate how it can be helpful in the process of identifying various aspects of the process of grasping of a situation. Our findings show that a suitably composed Concept Cartoon can be used to indicate understanding as well as misunderstanding and misconceptions. The second part of the text deals with various graphical representations of word problems, and their applicability in the process of solving unequal partition word problems. We show why schemes in the form of branched chains are not appropriate for representing the structure of this kind of word problems.

Résumé : Notre contribution est focalisée à la description des approches des enseignants du primaire en saisissant les situations basées sur les problèmes de type Parties-Tout. Dans la première partie, nous présentons un outil pédagogique appelé «Concept Cartoons» et nous le décrivons en tant qu'un outil d'aide pour identifier de divers phénomènes qui se manifestent en saisissant une situation. Nous constatons qu'un Concept Cartoon bien conçu peut être utilisé en tant qu'un indicateur de la compréhension, de la mauvaise compréhension et des erreurs. Dans la deuxième partie, nous proposons de diverses représentations graphiques (visualisations) des problèmes mathématiques et les possibilités de leur utilisation dans le processus de la résolution des problèmes de type Parties-Tout. Nous montrons les raisons pourquoi les schémas sous forme des „chaînes ramifiées" ne sont pas convenables pour la visualisation (représentation) de la structure de ce genre de problèmes.

Introduction

In the study presented here we focus on future primary school teachers, and consider the question of how to identify whether a future teacher grasps a situation related to a word problem with understanding (cf. Polya, 2004). Particularly we deal with unequal partition word problems. For our investigations we innovatively use an educational tool called Concept Cartoons.

Following up our research on problem posing presented at previous CIEAEMs (Tichá, 2009; Tichá & Hošpesová, 2013), we also consider the issue how using schemes for visual representation of the problem structure could help to grasp the situation related to the problem. We build on our recent experience showing that this approach might be helpful (Tichá, 2014; Tichá & Hošpesová, 2015).

Concept Cartoons

In mathematics education we make use of various schemes and visualizations. They help us to create a model of a problem or a record of its solution process. An educational tool called Concept Cartoons (CCs) can be also used for such purposes.

This tool was developed more than 20 years ago (Keogh & Naylor, 1993). Its original goal was to support teaching and learning in science classroom by generating discussion, stimulating investigation, and promoting learners' involvement and motivation (Naylor & Keogh, 2012). In later years the tool also expanded to other school subjects, including mathematics. Several years ago, the team of authors introduced a set of 130 CCs designed for classroom use in elementary school mathematics (Dabell, Keogh & Naylor, 2008).

Each Concept Cartoon (CC) is a picture presenting a situation well known to children, and a group of 5 children in a bubble-dialogue. The (mathematical) problem arises from the pictured situation, and sometimes is closely specified by the beginning of the text in the top left bubble – usually by the if-part of a conditional sentence. Texts in the other bubbles (and also the end of the text in the top left bubble) present alternative viewpoints on the situation and alternative solutions of the problem. One speech bubble is blank, with just "?" inside, to give a clear impression that there may be more alternative ideas that are not yet included in the dialog. See Fig. 1.

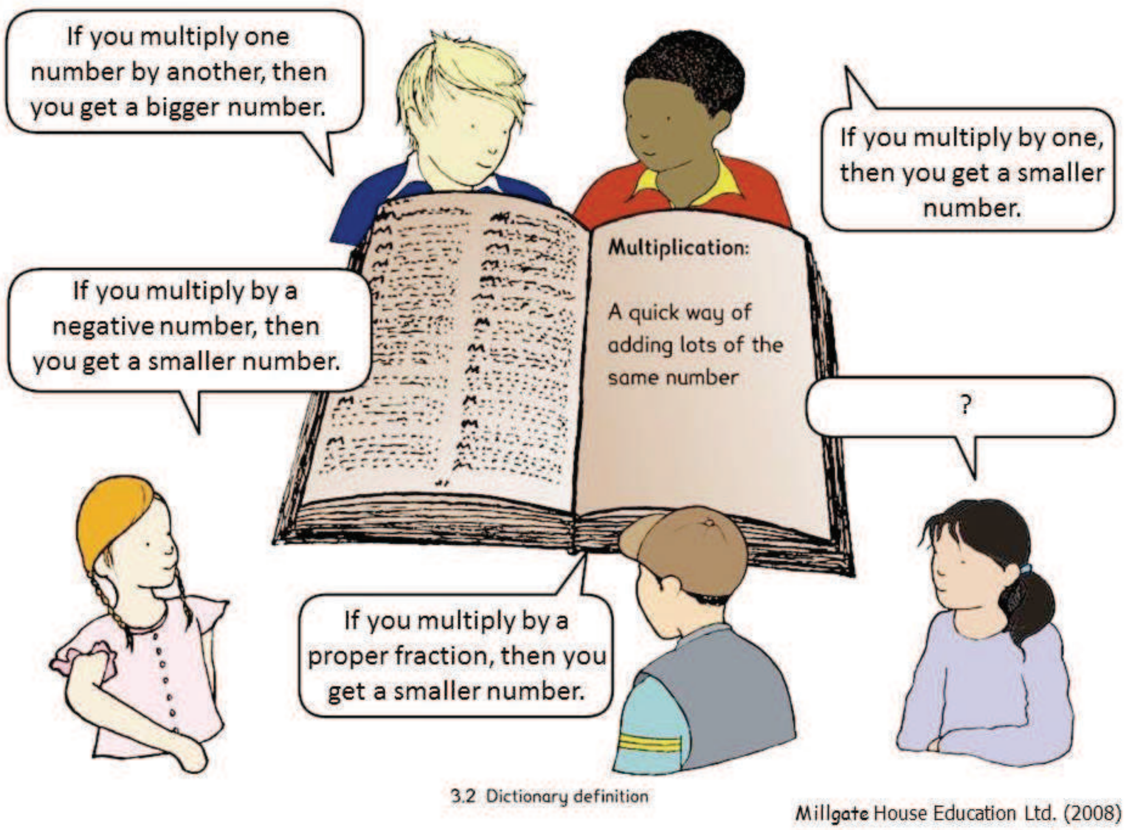


Figure 1: Concept Cartoon; taken from (Dabell, Keogh & Naylor, 2008), slightly modified.

The authors of the CC based the alternatives in bubbles on real classroom events or on common conceptions and misconceptions; they might also prepare some alternatives intentionally as authentic-looking unusual conceptions or misconceptions.

The situation pictured in the CC may be more or less open (with various ways of grasping, various ways of solving the problem based on the situation, or with multiple correct solutions to the problem – as in Fig. 1) or closed (with only one correct solution to the problem based on the situation – as in Fig. 2).

From the perspective of future primary school teachers' educators we feel the strength of CCs not only in teaching and learning, but also in diagnosing various types of teachers' mathematics knowledge: e.g. recently we have presented a study (Samková & Hošpesová, 2015) confirming that suitably chosen CCs allow to distinguish between subject matter knowledge and pedagogical content knowledge in the sense of Shulman (1986), and also between procedural and conceptual knowledge in the sense of Baroody, Feil and Johnson (2007). For that study we prepared a set of CCs, each of them presenting a closed situation leading to a calculation problem with one solution. We had two types of bubbles in these CCs: bubbles containing various procedures and their results, and bubbles containing just results. This combination of types of bubbles allowed us to investigate various aspect of teachers' knowledge (for details see Samková & Hošpesová, 2015).

Reported study (participants, methodology)

For the study reported here we chose a CC showing a closed situation leading to a calculation problem with one solution.

Since we did not plan to employ the blank bubble in this research, we replaced the "?" in the bottom left bubble by another alternative viewpoint, and offered there an intentionally prepared misconception. See Fig. 2.

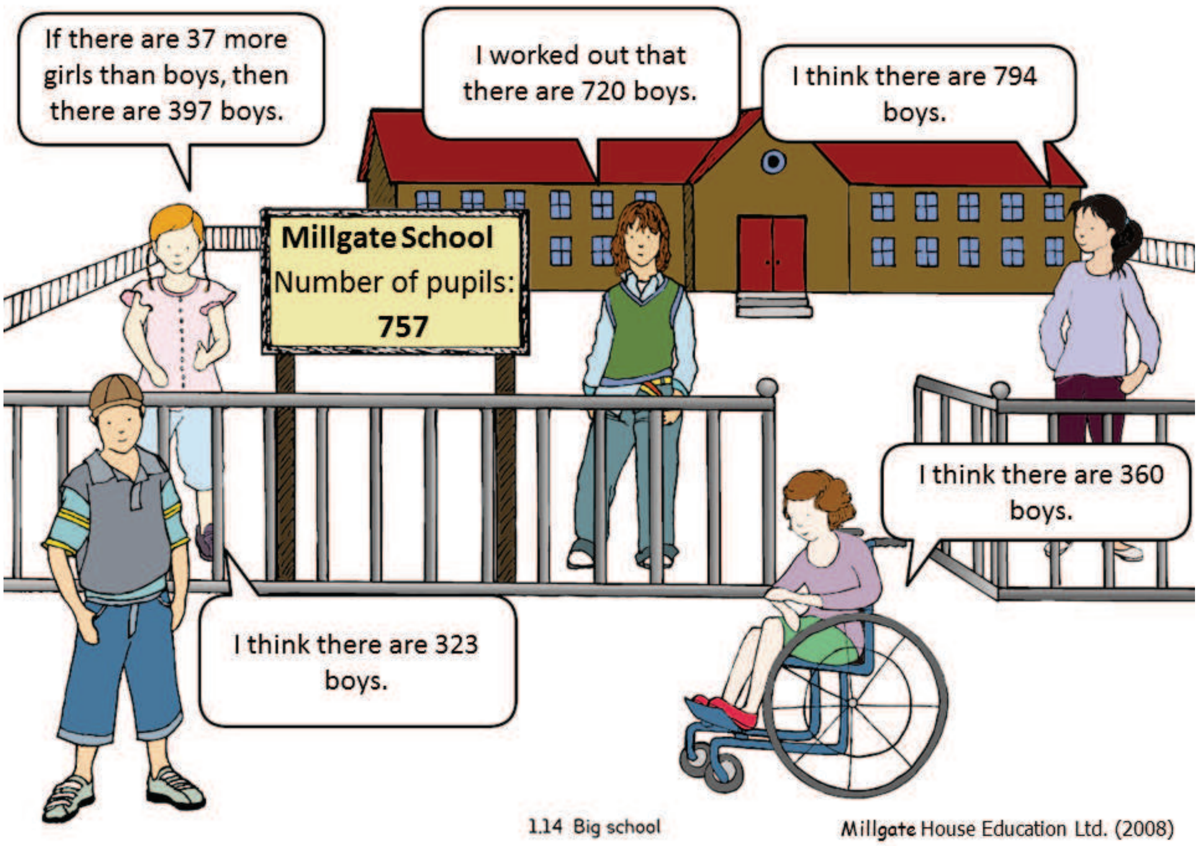


Figure 2: Concept Cartoon; taken from (Dabell, Keogh & Naylor, 2008), slightly modified.

Participants of our ongoing study are university master degree students – future primary school teachers. We collected data from them in two separate stages:

- In the first stage of the study we gave each participant a worksheet with the CC from Fig. 2. We asked them to decide which statements in bubbles are right, and to justify their decision. The participants worked individually, they wrote their conclusions on the worksheet. We put emphasis on the need to justify the decisions, in order to stimulate and deepen students' argumentation ability.
- In the second stage of the study we assigned the participants a word problem similar to the problem from the first stage, as a part of a standard written exam on arithmetic. We asked them not to use algebraic equations in the solution process, and to record the solution in detail.

The data from both stages of the study were processed qualitatively; we focused on aspects related to whether a participant grasps a situation with understanding. To be more precise, we monitored phenomena by which we characterized the process of *grasping of a situation* in our previous research (e.g. Tichá & Hošpesová, 2010), and phenomena which comply with the Polya's requirements for successful problem solution process (Polya, 2004).

By *grasping of a situation* we mean a process consisting of

- seeking and discovering the key phenomena of the situation and the relationships between them;
- insight into the subject of study;
- formulation of questions;
- searching for answers;
- interpretation of answers;
- evaluation of answers;
- identification of new questions and problems;
- continuing in a new process of searching using experience gained in previous activities

(Koman & Tichá, 1998).

Unequal partition problems

The problem outlined in the CC from Fig. 2 can be rephrased as a word problem:

> *There are 757 pupils in the Millgate School. Girls are 37 more than boys. How many boys are in the Millgate School?*

This word problem belongs to so-called unequal partition problems, i.e., problems of partition of a given quantity with the relationship between the parts expressed as a comparison of quantities (MacGregor & Stacey, 1998). In our case, both compared quantities are unknown.

In the second stage of the study we let students solve the following unequal partition word problem:

> *Tom and Carl have 68 marbles altogether. Carl has 14 marbles more than Tom. How many marbles has Tom?*

The situation with marbles is typical for unequal partition problems in our educational environment (cf. Novotná, 1997). These problems are usually solved either algebraically (i.e. using algebraic equations) or arithmetically (i.e. without equations, just by a sequence of arithmetic operations); graphical approach to the solution is not so common.

There are two prevailing arithmetic solution methods, based on two different representations of the situation: *sum-of-parts*, and *division-into-parts* (MacGregor & Stacey, 1998).

The case of *sum-of-parts* representation consists in searching one of the parts by taking away the extra quantity from the sum, and halving the reminder. In particular, solving the two word problems above results in calculating 757 – 37 = 720, 720 : 2 = 360 for the number of boys, and in calculating 68 – 14 = 54, 54 : 2 = 27 for the number of Tom's marbles. It is clearly seen from this representation that the unequal partition problem has a solution if the reminder is even, that means if the extra quantity and the sum have the same parity (both are even, or both are odd).

The case of *division-into-parts* representation consists in dividing the sum into two equal shares, and then adjusting these amounts by adding or subtracting half of the extra quantity. In particular, solving the two word problems above results in calculating 757 : 2 = 378.5, 37 : 2 = 18.5, 378.5 – 18.5 = 360 for the number of boys, and in calculating 68 : 2 = 34, 14 : 2 = 7, 34 – 7 = 27 for the number of Tom's marbles. We can see that if the solver can work with natural numbers only, then this method is not applicable for tasks with the extra quantity or the sum being odd.

Samples of actual findings

First stage of the study

i) Answers indicating understanding

Among the responses from the first stage of the study we revealed four different types of correct strategies:

- The most frequent one consisted in verifying (checking) of all offered alternatives. Such responses do not allow us to ascertain whether their authors know how to solve the problem and argue the solution procedure, but at least we can state that they grasped the situation successfully. Among these responses we revealed two different methods with diverse quality of the grasping process:
    - an analogue to guess-and-check method consisting of verifying every single offered alternative by calculating the number of girls and the number of all pupils for the given number of boys, and comparing such a number of all pupils with 757, e.g. by $397 + 37 = 434$, $397 + 434 = 831 \neq 757$ in case of the top left bubble;
    - a method using comparisons or estimates to reject immediately 794 and 720 for being too big.
- Less frequent was an arithmetic strategy consisting of a *sum-of-parts* method, i.e. of calculations $757 - 37 = 720$, $720 : 2 = 360$.
- Significantly fewer participants used another arithmetic strategy, a *division-into-parts* method, i.e. calculations $757 : 2 = 378.5$; $37 : 2 = 18.5$; $378.5 - 18.5 = 360$.
- Quite rarely appeared an algebraic strategy using an equation $x + (x + 37) = 757$, where x denotes the number of boys.

ii) Answers indicating misconceptions or misunderstanding

In the first stage of the study we revealed three different types of incorrect answers:

(a) Statement 720 is right, because $720 + 37 = 757$.

(b) No statement is right, because $757 : 2 - 37 = 341.5$ does not appear in bubbles.

(c) Statement 323 is right, because $(323 + 37) + 397 = 757$.

All three types indicate unsuccessful attempts to grasp the situation. Key phenomena of the situation and relationships between them were not discovered properly, the results were not verified with respect to the task, and on top of that – the author of the second answer was not even surprised by a decimal number as a result for the number of persons. We find interesting the fact that all authors of incorrect answers tried to justify somehow their answers.

Besides, the (c) misconception is a nice illustration of "take all numbers from the task, and do something with them" strategy – numbers 37 and 397 come from the top left bubble, and 757 comes from the information plate in the centre of the picture. There has to be noted that 323 is the intentionally prepared misconception we added into the blank bubble instead the question mark. Our previous experience had indicated that such a misconception might occur, and this suspicion was confirmed.

Second stage of the study

In the second stage of the study, the respondents mostly solved the word problem by the sum-of-parts method, i.e. $68 - 14 = 54$, $54 : 2 = 27$.

Among the incorrect solutions, only misconceptions analogical to (a) and (b) appeared.

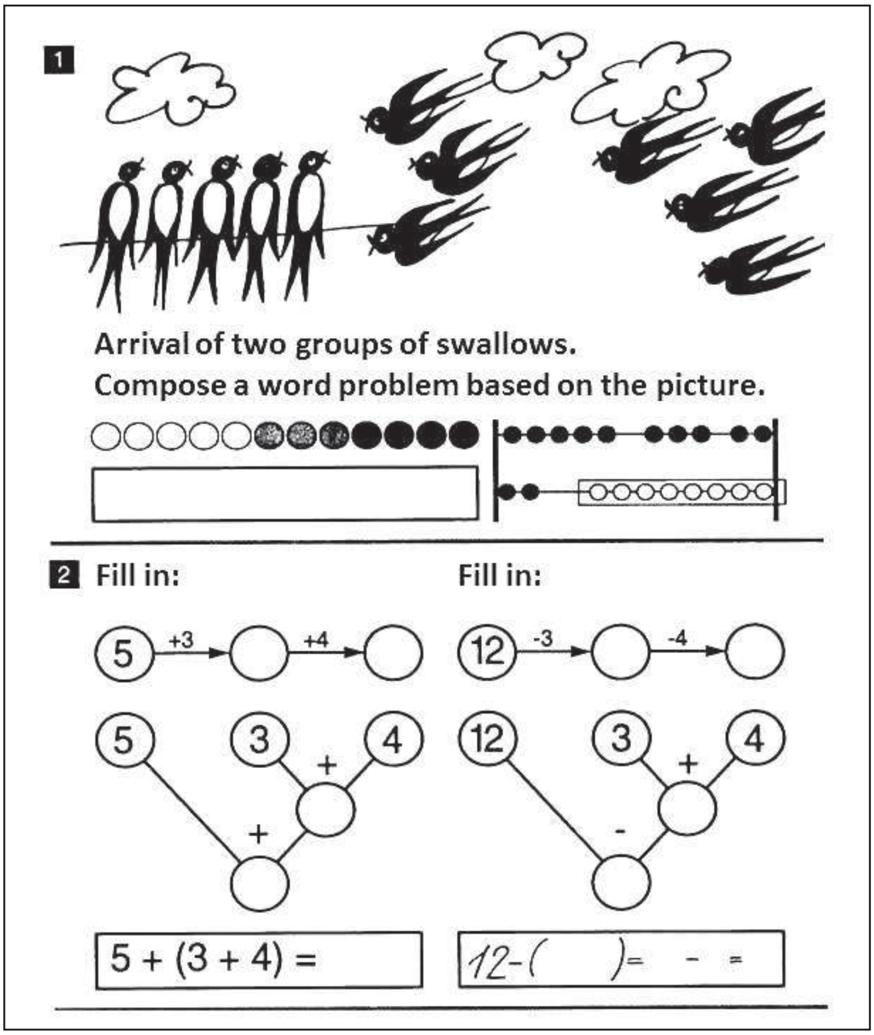
The student, who made the (c) misconception in the first stage, solved the task correctly by the sum-of-parts method in the second stage. That attracted our attention, and we interviewed this student subsequently to realize that she had just learned the sum-of-parts method by rote, without understanding. In this particular case, CCs helped us to reveal a weakness in understanding which could not be revealed in the standard written exam.

Graphical representation, visualization

To our surprise, none of the participants provided a graphical solution, nor offered a visualization of the problem – despite the fact that they had already met with schemes of problem structure in math courses.

We see the issue of graphical solutions and visualisations as important, since we believe that understanding deepens through enriching the repertoire of representations (cf. Janvier, 1987). We consider an iconic representation provided by visualization as very valuable, and as a non-skippable component of the process of grasping.

In our conception, schemes of a problem structure are linear or branched chains in the sense of Kittler and Kuřina (1994). Samples from their primary school textbook you can see in Fig. 3. We use these schemes as the means of visualization, as a graphical representation either of the problem structure or of the problem solving procedure. Moreover, we use them as a diagnostic tool in future



primary school teacher training (Tichá & Hošpesová, 2015).

Figure 3: Linear and branched chains as schemes of a problem structure; taken from (Kittler &

Kuřina, 1994, p. 70), translated.

Similar schemes were introduced also by Nesher and Hershkovitz for representing the problem structure: "Using a scheme, in our view, constitutes a mapping between semantic relations underlying a given text and its mathematical structure. The scheme serves as generalized habit of action in a given situation." (Nesher & Hershkovitz, 1994, p. 1)

For some types of word problems such schemes may help to grasp the situation, e.g. for two-step word problems such as:

> *There are 15 green and 17 blue matchbox cars on a big shelf, and 9 red matchbox cars on a small shelf. How many matchbox cars are there?*

For these word problems we can depict all key phenomena of the situation and relations between them to create two separate sub-schemes (Fig. 4 left). These sub-schemes have one common element and their composition produce a compound scheme suggesting us how to solve the problem (Fig. 4 right). For comparison see hierarchical scheme in (Nesher & Hershkovitz, 1994, p. 8).

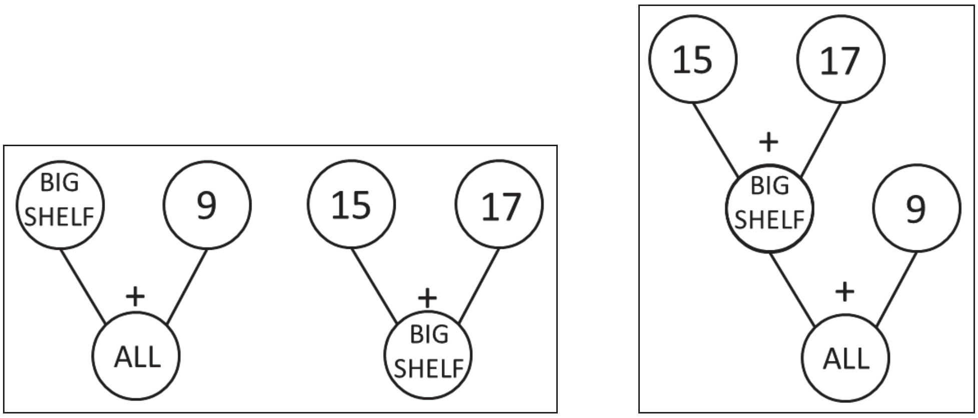


Figure 4: Schemes of the two-step problem with matchbox cars. Two separate sub-schemes (left), a compound scheme (right).

In the case of unequal partition problems the issue is more complicated. We may also create the sub-schemes (Fig. 5 left), but they have two common elements, and the compound scheme does not uncover the solution (in any rearrangement – see Fig. 5 middle, right). Concluded, in the case of unequal partition problems such schemes are not appropriate.

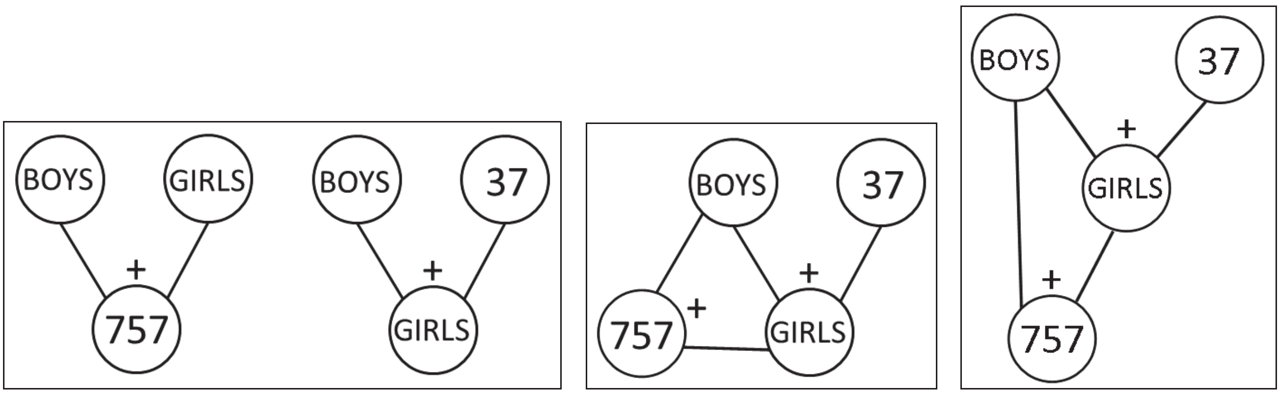
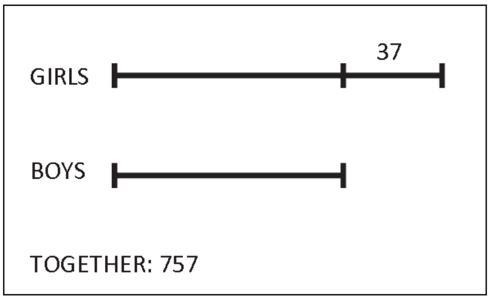


Figure 5: Schemes of the unequal partition problem outlined in Fig. 2. Two separate sub-schemes (left), a compound scheme (middle), a rearranged compound scheme (right).

For unequal partition problems we have to use a different kind of graphical representation. A suitable visualization can be obtained e.g. by a *segment model* (Novotná, 1997). This model serves just for getting an idea of the situation, thus the ratios of lengths of the segments are not supposed to correspond to the ratios of the numbers (Fig. 6 left). But our classroom experience show that students prefer to replace segments by rectangles, in order to be able to inscribe numbers inside, i.e. they use a *bar diagram* (Fig. 6 right).



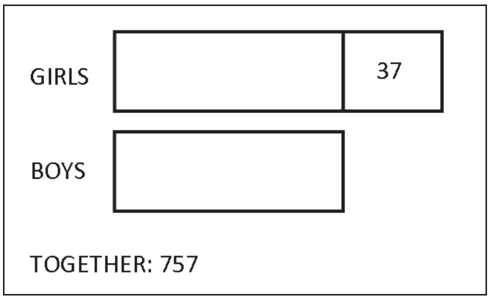


Figure 6: A segment model of the unequal partition problem outlined in Fig. 2 (left), a bar diagram of the same problem (right).

How to continue

As this text is a report of an ongoing study, we plan to continue in the research. We shall realize interviews with the participants of the study to reveal closer reasons for misconceptions that appeared in their responses, as well as reasons why none of them got use of a graphical representation.

As for CCs, preliminary findings of our study suggest that CCs could be used as a diagnostic tool for investigating future primary teachers' grasping of a situation. For the future we consider an interesting the issue of how particular features of CCs could help to reveal particular parts of the process of grasping.

We also plan to systematize CCs from the perspective of mathematics content. We aspire to create a set of CCs that would match Czech educational traditions, and cover regularly all important aspect of primary school mathematics.

Finally, this study shows how advantageous is the possibility to use a CC created by somebody else as a base for mediating our own thoughts and views. We may take such a CC, and adapt the content of its bubbles according to our intentions and previous experience. Our example with bottom left bubble in Fig. 2 illustrates how suitably chosen content of a newly added bubble can help to reveal a misconception that would stay unnoticed not only in the standard written exam, but probably also when working with the original version of the CC.

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